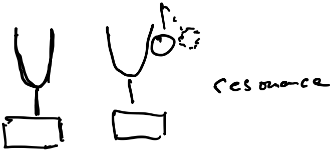
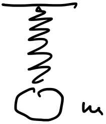
Complex dielectric constant and classical harmonic oscillator

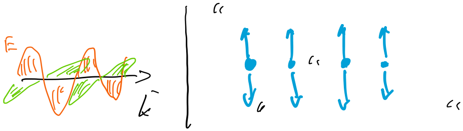
Here we need to answer the question, why light of different colours interacts with medium differently. What are the principles of this interactions?





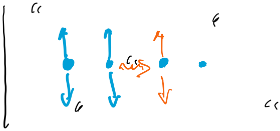
The interaction with light is going in 3 steps:

1) Local response of medium.
That means each local oscillator starts moving under the external e/m field.



2) Re-emission by secondary waves.
This part adds complexity! (Feynman)

Since charges start to move, they become the sources of secondary e/m wave.



3) Superposition of secondary waves.

We can imagine here, that original e/m wave is substituted by sum of secondary waves.

Maxwell's equations we already know:

$$\begin{cases} \bar{\nabla} \bar{D} = 0 \\ \bar{\nabla} \bar{B} = 0 \\ [\bar{\nabla} \bar{E}] = -\dfrac{\partial \bar{B}}{\partial t} \\ [\bar{\nabla} \bar{H}] = \dfrac{\partial \bar{D}}{\partial t} \end{cases}$$

Material equation: $\bar{D} = \varepsilon_0 \varepsilon \bar{E}$

ε - relative dielectric susceptibility, also known as dielectric constant.

$\bar{B} = \mu_0 \bar{H}$ $(\mu = 1)$

It is important to notice, we are considering media, where $\mu = 1$. That means we are going to discuss paramagnetic and diamagnetic media, but not ferromagnetic $(\mu > 1)$

Let's take plane waves!

$$\begin{cases} \bar E = \bar E_0 e^{i(\omega t - \bar k \bar r)} \\ \bar B = \bar B_0 e^{i(\omega t - \bar k \bar r)} \end{cases}$$

Since in our experiment we excite one frequency

$\Rightarrow$ monochromatic

$\omega = \mathrm{const}$

Little reminder

$\bar\nabla \Rightarrow -i\bar k$

$\frac{\partial}{\partial t} \Rightarrow i\omega$

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① Using M and material equation

$-i\bar k \bar D = 0$ $\quad \bar k \bar E = 0$ (through first material equation)

Other Maxwell's equations:

$-i[\bar k \bar E] = -i\omega \bar B$ $\quad\quad -i[\bar k \bar H] = i\omega \bar D$

We can simplify: $\quad (\bar B = \mu_0 \bar H)$

$$[\bar{k}\bar{E}] = \omega\bar{B} \qquad -[\bar{k}\bar{B}] = \underbrace{\mu_0\varepsilon_0}_{1/c^2}\varepsilon\omega\bar{E} = \frac{\omega}{c^2}\varepsilon\bar{E}$$

We will multiply by $\bar{k}$

$$[\bar{k}[\bar{k}\bar{E}]] = \omega[\bar{k}\bar{B}]$$

$$\underset{\overset{\shortparallel}{0}}{\bar{k}(\bar{k}\bar{E})} - k^2\bar{E} = -\frac{\omega^2}{c^2}\varepsilon\bar{E}$$

Hence we will have:

$$\boxed{\left(k^2 - \frac{\omega^2}{c^2}\varepsilon\right)\bar{E} = 0}$$

The only non-trivial solution is when

$$k^2 - \frac{\omega^2}{c^2}\varepsilon = 0$$

$\bar{E} = 0$ is not interesting for us.

$$\Downarrow$$

$$\boxed{k = \frac{\omega}{c}\sqrt{\varepsilon}}$$

1) If we have transparent dielectric, then

$\varepsilon > 0$ is real number.

In this case:

$\sqrt{\varepsilon} = n$ → we will call it $\underbrace{\text{refractive index}}$

In fact

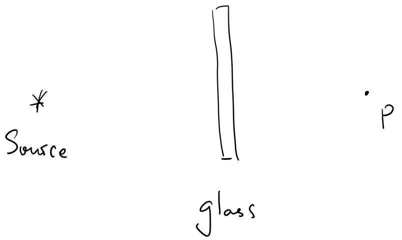
$$\sqrt{\varepsilon} = n(\omega) \Rightarrow k = \frac{\omega}{c}n = \frac{\omega}{v}$$

To remind, in vacuum $k = \frac{\omega}{c}$

$n = \frac{c}{v}$ – in vacuum

– in medium

Feynman approach (On origin of refractive index)

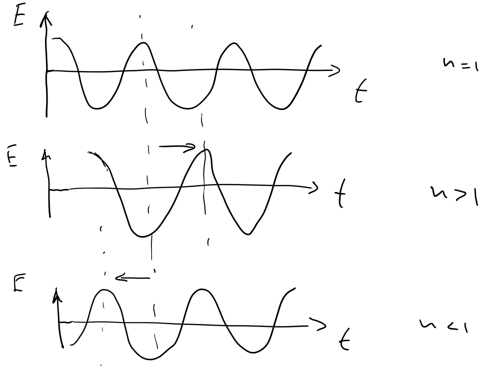


$E_p = E_s + \sum E_i$

E_i – field of each charge

$$E_s = E_0 \cos\omega\left(t - \frac{z}{c}\right) = E_0 e^{i\omega\left(t - \frac{z}{c}\right)}$$

$$E_p = E_0 e^{i\omega\left[t - (n-1)\frac{\Delta z}{c} - \frac{z}{c}\right]} = \underbrace{e^{i\omega(n-1)\frac{\Delta z}{c}}}_{\text{phase / delay}} E_0 e^{i\omega\left(t - \frac{z}{c}\right)}$$



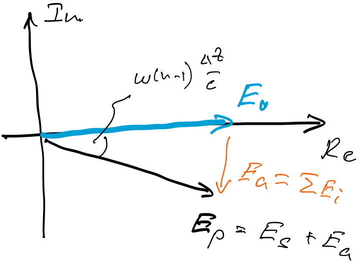
Instead of finding $E_p = E_s + \sum E_i$ we found the effect of plate is multiplying field by factor which shifts its phase. It is alright as we get same result by adding complex number.

If $\Delta z \to 0$ $\quad e^x = 1+x$, $x \to 0$

$$e^{-i\omega(n-1)\frac{\Delta z}{c}} \approx 1 - i\omega(n-1)\frac{\Delta z}{c}$$

$$\Downarrow$$

$$E_p = \underbrace{E_0 e^{i\omega(t-\frac{z}{c})}}_{E_s} - \underbrace{\frac{i\omega(n-1)\Delta z}{c} E_0 e^{i\omega(t-\frac{z}{c})}}_{\sum E_i}$$



2) If we have absorbing dielectric, then ε is the complex number

$$\sqrt{\varepsilon} = n'(\omega) \rightarrow \text{complex refractive index}$$

Let's try to derive it from classical Newton's mechanics

Model:

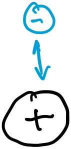
1) External e/m wave acts on neutral atom.
2) Atom becomes oscillating dipole.
3) This oscillating dipole emits e/m wave.

What would be the model of medium in which such e/m wave will propagate?

- isotropic
- homogeneous
- low density
- linear

We will discuss each of this adjectives as we will explore the theory and use each of them later.

What forces are acting on electron?



„+" is too heavy => do not move

e/m wave only oscillate „–", i.e. electron.

1) External driving force

$$\bar{F} = -e\bar{E}'$$

e – electron charge

Here $\bar{E}' = \bar{E} + \bar{E}_{in}$

$\bar{E}$ – external e/m wave.

$\bar{E}_{in}$ – internal field of other dipoles.

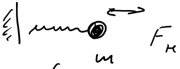
It is important here to consider low density medium => $\bar{E}_{in} = 0$

dipoles are too distant.

2) Quasi-elastic force

$$\bar{F}_{Qe} = -k\bar{r}$$

It should remind you of Hooke's force



$\bar{r}$ – displacement

Why $\bar{r}$ is power 1? This is approximation and thus the system is considered linear.

Linear vs Nonlinear optics.

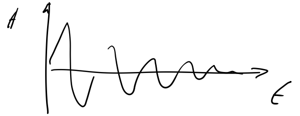
3) Radiation reaction force (Abraham - Lorentz force)

(analogue of viscosity friction).

$$F_{RF} = -\gamma \dot{r}$$

Why do we need to consider this force?

We have decaying oscillator, thus its amplitude decrease with time!



In classical mechanics we had to consider friction forces that are proportional to the body's speed.

Since we need to describe energy losses by oscillator, we formally consider radiation reaction force.

Thus, second Newton's law:

$$m\ddot{\bar{r}} = -k\bar{r} - \gamma\dot{\bar{r}} - e\bar{E}$$

$$\ddot{\bar{r}} + 2\beta\dot{\bar{r}} + \omega_0^2\bar{r} = -\frac{e}{m}\bar{E}$$

$$\begin{cases} 2\beta = \frac{\gamma}{m} \\ \omega_0^2 = \frac{k}{m} \end{cases}$$

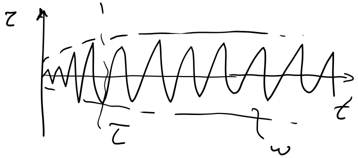
Since our field is harmonic wave:

$$\bar{E} = \bar{E}_0 e^{i\omega t}$$

There is no $k\bar{r}$, as atom is much smaller than λ, thus we can assume it is fully in the field of $\bar{E} = \bar{E}_0 e^{i\omega t}$ → dipole approximation

We will consider now equilibrated system

$t >> \tau$ τ is the time of equilibration



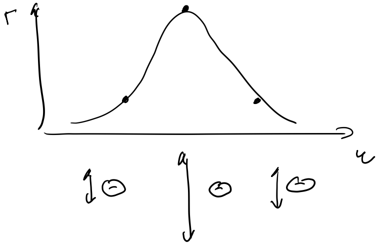
Once system is equilibrated, the oscillator will have frequency ω.

Second Newton's law will result in solution:

$$\bar{r}(\omega) = \frac{-\frac{e}{m}\bar{E}(\omega)}{\omega_0^2 - \omega^2 + 2i\beta\omega}$$

What does it mean?

The displacement $\bar{r}$ depends on how ω is different from ω_0. This is called resonance.



Atom now has the dipole moment $\bar{p} = -e\bar{r}$

$$\bar{p}(\omega) = \frac{e^2/m \, \bar{E}(\omega)}{\omega_0^2 - \omega^2 + 2i\beta\omega}$$ – aslo function of ω.

Let's N – concentration of the atoms.

$N = \text{const} \Rightarrow$ homogeneous medium

If $N \neq \text{const}$, then it will be turbid medium, i.e. medium, where refractive index at different spots is different $\Rightarrow$ scattering effects.

Medium polarization:

$$\bar{P} = N\bar{p}$$ $\rightsquigarrow$ dipole moment of single atom

$P = \varepsilon_0 \chi \bar{E}$ $\quad$ χ – dielectric susceptibility

We will only consider χ bieng scalar, i.e. $\bar{P} \parallel \bar{E}$. This means the medium is isotropic.

For anisotropic media χ is tensor.

Hence

$$\chi = \frac{Ne^2}{m\varepsilon_0} \frac{1}{\omega_0^2 - \omega^2 + 2i\beta\omega}$$

In this case:

$$\bar{D} = \varepsilon_0 \bar{E} + \bar{P} = \varepsilon_0 \bar{E} + \varepsilon_0 \chi \bar{E} = \varepsilon_0 \underbrace{(1+\chi)}_{\varepsilon} \bar{E}$$

The dielectric constant:

$$\boxed{\varepsilon(\omega) = 1 + \chi = 1 + \frac{Ne^2}{m\varepsilon_0} \frac{1}{\omega_0^2 - \omega^2 + 2i\beta\omega}}$$

Let's summarize:

1) $\varepsilon = f(\omega)$

2) $\varepsilon(\omega)$ is complex function.

β is responsible for decay.

When there is no absorption $\Rightarrow$ no energy losses $\Rightarrow$

$\Rightarrow$ no decay $\beta = 0 \Rightarrow \varepsilon(\omega)$ is real number.

For absorbing media:

$n' = \sqrt{\varepsilon} = n - iK$

n - refractive index

K - decaying index

damping index

$$\varepsilon = n^2 - K^2 - 2inK$$

$$\begin{cases} n^2 - K^2 = 1 + \dfrac{Ne^2}{m\varepsilon_0} \dfrac{\omega_0^2 - \omega^2}{(\omega_0^2-\omega^2)^2 + 4\beta^2\omega^2} \longrightarrow \mathrm{Re}\,\varepsilon \\[2ex] nK = \dfrac{Ne^2}{m\varepsilon_0} \dfrac{\beta\omega}{(\omega_0^2-\omega^2)^2 + 4\beta^2\omega^2} \longrightarrow \mathrm{Im}\,\varepsilon \end{cases}$$

To simplify these equations we will need two assumptions:

1) When $\omega \approx \omega_0$ (near the resonance):

$$\omega \approx \omega_0$$

$$(\omega_0^2-\omega^2)^2 = (\omega_0+\omega)(\omega_0-\omega) = 2\omega_0 \Delta\omega$$

2) We see, that n and κ are strongly intertwined. To decouple them, we can again take into consideration low density.

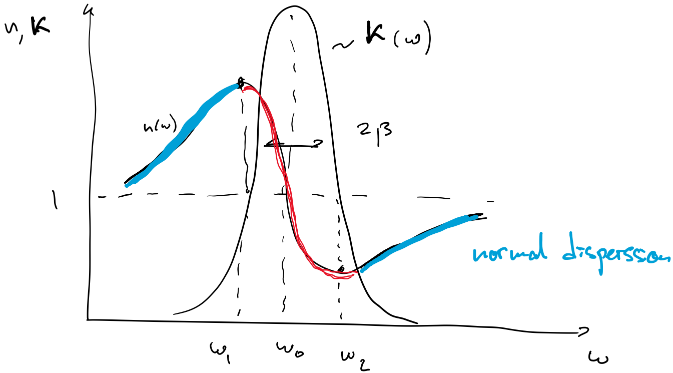
Taking into account first assumption:

$$\begin{cases} n^2-\kappa^2 = 1 + \dfrac{Ne^2}{m\varepsilon_0} \cdot \dfrac{1}{2\omega_0} \cdot \dfrac{\Delta\omega}{\Delta\omega^2+\beta^2} \\[2ex] n\kappa = \dfrac{Ne^2}{m\varepsilon_0} \cdot \dfrac{1}{4\omega_0} \cdot \dfrac{\beta}{(\Delta\omega)^2+\beta^2} \end{cases}$$

Since N is small $n\kappa \ll 1$. Thus:

$$n^2-\kappa^2 \approx 1.$$

$$n \approx \sqrt{\varepsilon} \approx 1 + \frac{1}{2} \cdot \frac{Ne^2}{m\varepsilon_0} \cdot \frac{1}{2\omega_0} \cdot \frac{\Delta\omega}{\Delta\omega^2+\beta^2}$$

$$\begin{cases} n(\omega) \approx 1 + \dfrac{Ne^2}{m\varepsilon_0} \cdot \dfrac{1}{4\omega_0} \dfrac{\Delta\omega}{(\Delta\omega)^2 + \beta^2} \\[2em] \kappa(\omega) \approx \dfrac{Ne^2}{m\varepsilon_0} \cdot \dfrac{1}{4\omega_0} \dfrac{\beta}{(\Delta\omega)^2 + \beta^2} \end{cases}$$